# A GEVREY MICROLOCAL ANALYSIS OF MULTI-ANISOTROPIC DIFFERENTIAL OPERATORS

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Abstract. We give a microlocal version of the theorem of iterates in multi-anisotropic Gevrey classes for multi-anisotropic hypoelliptic differential operators.

## 1. Introduction

A fundamental result of Gevrey microlocal regularity due to Hörmander [13] is

$$WF_s(u) \subset WF_s(P(x,D)u) \cup Char(P) \ , \tag{1.1}$$

where $P(x,D)$ denotes a differential operator with analytic coefficients in $\Omega$, $Char(P)$ its set of characteristic points $(x,\xi) \in \Omega \times \mathbb{R}^n$ and $WF_s(u)$ is the Gevrey wave front of the distribution $u \in \mathfrak{D}'(\Omega)$.

Let $WF_s(u, P(x,D))$, see [2], be the Gevrey wave front of the distribution $u \in \mathfrak{D}'(\Omega)$ with respect to the iterates of the operator $P(x,D)$, then the result (1.1) is made more precise by the following inclusion

$$WF_s(u) \subset WF_s(u, P(x,D)) \cup Char(P) \ , \tag{1.2}$$

since

$$WF_s(u, P(x,D)) \subset WF_s(P(x,D)u) \ . \tag{1.3}$$

Various extensions and generalizations of results (1.1) and (1.2) have been obtained, according as one considers the classes of elliptic or hypoelliptic differential operators or one considers different notions of homogeneity associated to these classes of operators, see e. g. [2], [3], [8], [15], [16] and [17].

In [14] a microlocal analysis of the so called inhomogeneous Gevrey classes [15], see also [6], has been introduced. The $\varphi-$inhomogeneous Gevrey wave front of a distribution $u \in \mathfrak{D}'(\Omega)$, denoted $WF_\varphi(u)$, is defined with respect to a weight function $\varphi$.

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The method of Newton's polyhedron, see [9] or [1], permits to approach differential operators with respect to their multi-quasihomogeneity. In this situation the $\varphi-$inhomogeneous Gevrey wave front $WF_{\varphi}(u)$ is characterized by a weight $\varphi$ equals to the function $|\xi|_{\mathbb{P}}$ defined by the Newton's polyhedron $\mathbb{P}$ of the operator $P(x,D)$ and it is denoted by $WF_{s,\mathbb{P}}(u)$.

An interpretation of the $\varphi-$inhomogeneous Gevrey microlocal analysis to the multi-anisotropic case is given in the paper [8], where a theorem in the spirit of the result (1.1) for a class of multi-quasihomogeneous hypoelliptic differential operators is obtained.

The aim of this paper is to obtain a result in the spirit of (1.2) for a class of multi-anisotropic hypoelliptic differential operators including the classes of operators studied in [2], [4], [7], [8], [10], [16] and [17]. The section 2 is an adapted modification of the $\varphi-$inhomogeneous Gevrey wave front of Liess-Rodino, see [14] and [8], to our multi-anisotropic case in the spirit of [13] and [16]. In section 3 we introduce and study the multi-anisotropic Gevrey wave front with respect to the iterates of an operator $P(x,D)$ and its Newton's polyhedron $\mathbb{P}$, denoted $WF_{s,\mathbb{P}}(u,P(x,D))$, the following section 4 gives the microlocal result of type (1.2) for the studied class of differential operators. This class is microlocally characterized by the following definition.

**Definition 1.** *Let $x_0\in\Omega,\xi_0\in\mathbb{R}^n\backslash\{0\}$ and $P(x,D)$ be a differential operator with coefficients in the anisotropic Gevrey class $G^{s,q}(\Omega)$, we say $(x_0,\xi_0)\notin\sum_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}(P)$ if there exists an open neighbourhood $U\subset\Omega$ of $x_0$, an open $q$-quasiconic neighbourhood $\Gamma\subset\mathbb{R}^n\backslash\{0\}$ of $\xi_0$ and a constant $c>0$ such that $\forall\,(x,\xi)\in U\times\Gamma$,*

$$\begin{cases} |\xi|_{\mathbb{P}}^{\mu'}\leq c\,|P(x,\xi)|\,, \\ \left|D_x^{\alpha}D_{\xi}^{\beta}P(x,\xi)\right|\leq c^{|\alpha|+1}<\alpha,q>^{s\mu<\alpha,q>}|P(x,\xi)|\,|\xi|_{\mathbb{P}}^{\delta|\alpha|-\rho|\beta|}\,,\end{cases}$$

*where the numbers $\rho,\delta,\mu'$ and $\mu$ satisfy $0\leq\delta<\rho\leq1$ and $\delta\mu<\mu'\leq\mu$.*

The principal result of this work is the following theorem.

**Theorem 1.** *Let $u\in\mathfrak{D}'(\Omega)$, $P(x,D)$ a differential operator with coefficients in $G^{s,q}(\Omega)$ and $\rho,\delta,\mu',\mu$ such that $0\leq\delta<\rho\leq1$ and $\delta\mu<\mu'\leq\mu$, then*

$$WF_{s',\mathbb{P}}(u)\subset WF_{s,\mathbb{P}}(u,P)\ \cup\sum\nolimits_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}(P), \tag{1.4}$$

*where $s'=\max\left(\frac{s\mu}{\mu'-\delta\mu},\frac{s}{\rho-\delta}\right)$.*

## 2. Multi-anisotropic Gevrey wave front

This section is an adaptation with a slight modification of the inhomogeneous Gevrey microlocal analysis introduced in [14], see also [15] and [8], to the multi-anisotropic case.

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $P(x,D)$ be a linear partial differential operator with coefficients in $C^\infty(\Omega)$,

$$P(x,D)=\sum_{\alpha\in\Lambda} a_\alpha(x)\,D^\alpha\ ,$$

where $\Lambda$ is a finite subset of $\mathbb{Z}_+^n$.

**Definition 2.** *Let $x_0\in\Omega$, the Newton's polyhedron of the operator $P(x,D)$ at the point $x_0$, denoted $\mathbb{P}(x_0)$, is the convex hull of*

$$\{0\}\cup\left\{\alpha\in\mathbb{Z}_+^n, a_\alpha(x_0)\neq 0\right\}\ .$$

**Remark 1.** *A Newton's polyhedron $\mathbb{P}$ is always characterized by*

$$\mathbb{P}=\underset{a\in\mathcal{A}}{\cap}\left\{\alpha\in\mathbb{R}_+^n, <\alpha,a>\leq 1\right\}\ ,$$

*where $\mathcal{A}(\mathbb{P})$ is a finite subset of $\mathbb{R}^n$.*

**Definition 3.** *The Newton's polyhedron $\mathbb{P}$ is said to be regular if for any $a=(a_1,...,a_j,...,a_n)\in\mathcal{A}$ we have $a_j>0,\forall j=1,..,n$.*

**Definition 4.** *The operator $P(x,D)$ is said regular if it satisfies the following conditions :*

1. $\mathbb{P}(x_0)=\mathbb{P}$, $\forall x_0\in\Omega$ .
2. $\mathbb{P}$ *is a regular polyhedron.*

**Remark 2.** *In this paper we consider only regular operators.*

Let $\mathbb{P}$ be a regular polyhedron, we set

$$\begin{aligned}
\mathcal{V}(\mathbb{P}) &= \left\{s^0=0,s^1,..,s^m\right\} \text{ the set of the vertices of } \mathbb{P}\ .\\
\mu_j &= \max a_j^{-1}\ ,\ a\in\mathcal{A}\ .\\
\mu &= \max \mu_j\ .\\
q &= \left(\frac{\mu}{\mu_1},..,\frac{\mu}{\mu_n}\right)\ .\\
k(\alpha) &= \inf\left\{t>0, t^{-1}\alpha\in\mathbb{P}\right\}=\max_{a\in\mathcal{A}}<\alpha,a>\ .\\
|\xi|_{\mathbb{P}} &= \left(\sum_{i=1}^m\left(\xi^{2s^i}\right)^{1/\mu}\right)^{1/2}\ .\\
|\xi|_q &= \left(\sum_{j=1}^n(\xi_j)^{2/q_j}\right)^{1/2}\ .
\end{aligned}$$

**Definition 5.** *Let $s \geq 1$ and $\mathbb{P}$ be a regular polyhedron, we denote $G^{s,\mathbb{P}}(\Omega)$ the space of functions $u \in C^{\infty}(\Omega)$ such that $\forall K$ compact of $\Omega$, $\exists C > 0, \forall \alpha \in \mathbb{Z}_{+}^{n}$,*

$$\sup_{K} |D^{\alpha} u| \leq C^{|\alpha|+1} k(\alpha)^{s\mu k(\alpha)} \quad . \tag{2.1}$$

**Example 1.** *If the operator $P$ is $l$-quasi-elliptic of order $m$, with the weight $l = (l_1, ..., l_n)$, so its Newton's polyhedron $\mathbb{P}$ is the simplex of vertices $\{0,\ m_j e_j,\ j = 1, .., n\}$, which is obviously regular. In this case the set $\mathcal{A}$ coincides with the vector $\sum_{j=1}^{n} m_j^{-1} e_j$, and we have $\mu_j = m_j$, $\mu = m$, $l = q = (\frac{m}{m_1}, .., \frac{m}{m_n})$. If $\alpha \in \mathbb{Z}_{+}^{n}$, then $k(\alpha) = m^{-1} < \alpha, q >$ and we obtain $G^{s,\mathbb{P}}(\Omega) = G^{s,q}(\Omega)$ the anisotropic Gevrey space, i.e. the space of functions $u \in C^{\infty}(\Omega)$ such that $\forall K$ compact of $\Omega$, $\exists C > 0, \forall \alpha \in \mathbb{Z}_{+}^{n}$,*

$$\sup_{K} |D^{\alpha} u| \leq C^{|\alpha|+1} \alpha_1!^{q_1} ... \alpha_n!^{q_n} \quad .$$

The following lemma, obtained in [8], gives the existence of a truncation sequence, following the fundamental lemma 2.2 of [13] in the multi-anisotropic case. The quasihomogeneous case is a result of [16, lemma 1.2].

**Lemma 1.** *Let $K$ be a compact set of $\mathbb{R}^n$ and let $s \geq 1$, then there exists a sequence $(\chi_N) \subset C_0^{\infty}(\mathbb{R}^n)$ such that $\chi_N = 1$ on $K$ and*

$$|D^{\alpha} \chi_N| \leq C \left(C N^{s\mu}\right)^{<\alpha, a>} \quad \text{if} < \alpha, a > \leq N, \forall a \in \mathcal{A},\ N = 1, 2, ... \quad . \tag{2.2}$$

A characterization of $G^{s,\mathbb{P}}(\Omega)$ using the Fourier transform is given by the following theorem.

**Theorem 2.** *Let $x_0 \in \Omega$ and $u \in \mathfrak{D}'(\Omega)$, then $u$ is $G^{s,\mathbb{P}}$ in a neighbourhood of $x_0$ if, and only if there exists a neighbourhood $U$ of $x_0$ and a sequence $(u_N)$ in $\mathcal{E}'(\Omega)$ such that*

*i) $u_N = u$ in $U$, $N = 1, 2, ...$ .*

*ii) $u_N$ is bounded in $\mathcal{E}'(\Omega)$ .*

*iii) $|\widehat{u_N}(\xi)| \leq C \left(\frac{CN^s}{|\xi|_{\mathbb{P}}}\right)^{\mu N}$, $N = 1, 2, ...$ .*

*Proof.* See [14] and [8]. □

We give now a microlocalization of the definition 5. It is an adapted modification , in the spirit of [13] and [16], of the $\varphi-$inhomogeneous Gevrey wave front of Liess-Rodino, see [14] and [8], to our multi-anisotropic case. It coincides exactly with the classical definition of the quasihomogeneous case.

**Definition 6.** *Let $x_0 \in \Omega$, $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $u \in \mathfrak{D}'(\Omega)$, we say that $u$ is $G^{s,\mathbb{P}}-$microregular at $(x_0,\xi_0)$, we denote $(x_0,\xi_0) \notin WF_{s,\mathbb{P}}(u)$, if there exists $C>0$, a neighbourhood $U$ of $x_0$ in $\Omega$, a q-quasiconic neighbourhood $\Gamma$ of $\xi_0$ in $\mathbb{R}^n \setminus \{0\}$ and a sequence $(u_N) \subset \mathcal{E}'(\Omega)$ such that*

*i) $u_N = u \quad in \quad U,\ N=1,2,...$ .*
*ii) $u_N$ is bounded in $\mathcal{E}'(\Omega)$ .*
*iii) $\left|\widehat{u_N}(\xi)\right| \le C\left(\frac{CN^s}{|\xi|_{\mathbb{P}}}\right)^{\mu N},\ N=1,2,...,\ \xi \in \Gamma$ .*

We recall that a subset $\Gamma \subset \mathbb{R}^n$ is said $q$-quasiconic if

$$\forall \xi \in \Gamma, \forall t>0:\ (t^{q_1}\xi_1,...,t^{q_n}\xi_n) \in \Gamma.$$

**Remark 3.** *The definition 6 coincides exactly with the quasihomogeneous case, see* [16], *if the polyhedron $\mathbb{P}$ is the simplex of vertices $\{0,\ m_j e_j,\ j=1,..,n\}$.*

Using the truncation sequence $(\chi_N)$ we obtain the following lemma, see [8].

**Lemma 2.** *Let $u \in \mathfrak{D}'(\Omega)$ and $(x_0,\xi_0) \notin WF_{s,\mathbb{P}}(u)$ and let $U,\Gamma$ be as in definition 6. If $K$ is a compact neighbourhood of $x_0$ in $U$, $F$ is a q-quasiconic compact neighbourhood of $\xi_0$ in $\Gamma$ and $(\chi_N) \subset C_0^\infty(U)$ equal to 1 on $K$ satisfying (2.2), then there exists $p_0 \in \mathbb{Z}_+, N_0 \in \mathbb{Z}_+$ such that the sequence $(\chi_{p_0 N+N_0} u)$ satisfies i)-iii) in $K$ and $F$.*

We define the $G^{s,\mathbb{P}}-$singsupp$(u)$ as the complementary of the biggest open subset of $\Omega$ where $u$ is $G^{s,\mathbb{P}}$. The relation between the multi-anisotropic Gevrey wave front and the multi-anisotropic Gevrey singular support is given by the following proposition.

**Proposition 1.** *Let $u$ be a distribution in $\Omega$, then the projection of $WF_{s,\mathbb{P}}(u)$ on $\Omega$ is the $G^{s,\mathbb{P}}-$singsupp$(u)$.*

*Proof.* It follows the similar proof of [8]. □

The microlocal property of the differential operator $P(x,D)$ with respect to the $G^{s,\mathbb{P}}-$wave front $WF_{s,\mathbb{P}}(u)$ is given by the following theorem.

**Theorem 3.** *Let $u \in \mathfrak{D}'(\Omega)$ and $P(x,D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$, then*

$$WF_{s,\mathbb{P}}(Pu) \subset WF_{s,\mathbb{P}}(u). \tag{2.3}$$

*Proof.* See [14] and [15]. □

**Remark 4.** *The product of two functions of the space $G^{s,\mathbb{P}}(\Omega)$ does not belong in general to $G^{s,\mathbb{P}}(\Omega)$, but if $g \in G^{s,q}(\Omega)$ and $f \in G^{s,\mathbb{P}}(\Omega)$, then $gf \in G^{s,\mathbb{P}}(\Omega)$, see* [11]. *This justifies the optimal choice of the regularity of the coefficients of the operator $P(x,D)$.*

## 3. Multi-anisotropic Gevrey wave front with respect to the iterates of a differential operator

The Gevrey microlocal analysis with respect to the iterates of a differential operator has been introduced for the first time by P. Bolley and J. Camus in [2] in the homogeneous case. L. Zanghirati in [16] has adapted it to the quasihomogeneous case. The aim of this section is to extend this analysis to the multi-quasihomogeneous case.

**Definition 7.** *Let $r \in \mathbb{R}$ and $s \geq 1$, we denote $G_r^{s,\mathbb{P}}(\Omega, P)$ the space of distributions $u \in \mathfrak{D}'(\Omega)$ such that $\forall K$ compact of $\Omega, \exists C > 0, \forall N \in \mathbb{Z}_+$,*

$$\left\|P^N u\right\|_{H^r(K)} \leq C\left(CN^s\right)^{\mu N}.$$

*The space of Gevrey vectors of the operator $P$ is by definition*

$$G^{s,\mathbb{P}}(\Omega, P) = \underset{r\in\mathbb{R}}{\cup} G_r^{s,\mathbb{P}}(\Omega, P).$$

The space of Gevrey vectors $G^{s,\mathbb{P}}(\Omega, P)$ of the operator $P$ is described with the help of the Fourier transform in the following lemma.

**Lemma 3.** *Let $x_0 \in \Omega$ and $u \in \mathfrak{D}'(\Omega)$, then $u \in G^{s,\mathbb{P}}(V, P)$ for a neighbourhood $V$ of $x_0$ if, and only if, there exists a neighbourhood $U$ of $x_0, U \subset V$, $C > 0$, $M \in \mathbb{R}$ and a sequence $(f_N)$ in $\mathcal{E}'(V)$ such that*
*l) $f_N = P^N u$ in $U$, $N = 0, 1, \ldots$ .*
*ll) $\left|\widehat{f_N}(\xi)\right| \leq C\left(CN^s\right)^{\mu N}(1+|\xi|)^M$, $\xi \in \mathbb{R}^n, N = 0, 1, \ldots$ .*

*Proof.* It follows the proof of proposition 1.4 of [2]. □

The following technical lemma is important for the sequel.

**Lemma 4.** *Let $K$ be a compact subset of $\Omega$ and $(\chi_N)$ a sequence in $C_0^\infty(\mathbb{R}^n)$ satisfying (2.2) and $\mathfrak{F}$ a subset of $G^{s,q}(\Omega)$ such that*

$$\exists C > 0,\ \forall a \in \mathcal{A},\ \forall v \in \mathfrak{F},\ \sup_K |D^\alpha v| \leq C\left(C < \alpha, a >^s\right)^{\mu<\alpha,a>},$$

*then $\exists C_1 > 0$, $\forall v_1, .., v_{j-1} \in \mathfrak{F}$, $\forall \alpha^1, .., \alpha^j \in \mathbb{Z}_+^n, \forall a^1, .., a^j \in \mathcal{A}$, $< \alpha^1, a^1 > +..+ < \alpha^j, a^j > \leq N$, we have*

$$\begin{aligned} &\sup_K \left|D^{\alpha^1} v_1 D^{\alpha^2} v_2 ... D^{\alpha^{j-1}} v_{j-1} D^{\alpha^j} \chi_N\right| \\ \leq\ & C_1^{N+1}\left(\left(< \alpha^1, a^1 >\right)^{\mu<\alpha^1,a^1>}\left(< \alpha^2, a^2 >\right)^{\mu^2<\alpha^2,a^2>} ... \left(< \alpha^j, a^j >\right)^{\mu^2<\alpha^j,a^j>}\right)^s. \end{aligned}$$

*Proof.* It is sufficient to see at first that $G^{s,q}(\Omega) \subset G^{s,\mu a}(\Omega), \forall a \in \mathcal{A}$ and for any $a, b \in \mathcal{A}$, we have

$$(\mu < \alpha, a >)^{(\mu<\alpha,a>)} \leq \left(\mu^2 < \alpha, b >\right)^{\left(\mu^2<\alpha,b>\right)},\ \alpha \in \mathbb{Z}_+^n,$$

and apply after lemma 2.3 of [16] or adapt lemma 5.3 of [13]. □

Thanks to the truncation sequence $(\chi_N)$, if $u \in \mathfrak{D}'(\Omega)$, the sequence $u_N = \chi_N u$ is bounded in $\mathcal{E}'(\Omega)$ and then $\exists C > 0$, $|\widehat{u_N}(\xi)| \leq C(1+|\xi|)^M$, $\xi \in \mathbb{R}^n, N \in \mathbb{Z}_+$. In the problem of iterates this property is precised by the following result.

**Lemma 5.** *Let $K$ be a compact subset of $\Omega$ and let $(\chi_N)$ be a sequence in $C_0^\infty(K)$ satisfying (2.2), then $\forall u \in \mathfrak{D}'(\Omega), \exists p_0 > 0, \forall p > p_0, \forall r \in \mathbb{Z}_+$, the sequence $f_N = \chi_{pN+r}P^N$ satisfies*

$$\widehat{f_N}(\xi) \leq C\left(C\left(N^{s\mu} + |\xi|_{\mathbb{P}}\right)\right)^{\mu N+M},\ \xi \in \mathbb{R}^n, N \in \mathbb{Z}_+, \tag{3.1}$$

*Proof.* It does not differ substantially from its quasihomogeneous similar lemma 2.4 of [16]. □

The belonging to the space $G^{s,\mathbb{P}}(\Omega, P)$ is microlocally characterized by the following definition.

**Definition 8.** *Let $u \in \mathfrak{D}'(\Omega)$, $(x_0,\xi_0) \in \Omega \times \mathbb{R}^n\backslash\{0\}$ and $P(x,D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$. We say that $u$ is $G^{s,\mathbb{P}}-$microregular with respect to the iterates of $P(x,D)$ at $(x_0,\xi_0)$, we denote $(x_0,\xi_0) \notin WF_{s,\mathbb{P}}(u,P)$, if there exists $C > 0, M \in \mathbb{R}$, a neighbourhood $U$ of $x_0$ in $\Omega$, a q-quasiconic neighbourhood $\Gamma$ of $\xi_0$ in $\mathbb{R}^n\backslash\{0\}$ and a sequence $(f_N) \subset \mathcal{E}'(\Omega)$ such that*

*j)* $f_N = P^N u$ *in* $U$, $N \in \mathbb{Z}_+$ .

*jj)* $\left|\widehat{f_N}(\xi)\right| \leq C\left(C\left(N^{s\mu} + |\xi|_{\mathbb{P}}\right)\right)^{\mu N+M},\ \xi \in \mathbb{R}^n, N \in \mathbb{Z}_+$ .

*jjj)* $\left|\widehat{f_N}(\xi)\right| \leq C\left(CN^s\right)^{\mu N}(1+|\xi|)^M,\ \xi \in \Gamma, N \in \mathbb{Z}_+$ .

The following proposition gives the link between the $G^{s,\mathbb{P}}-$singularities of a distribution $u \in \mathfrak{D}'(\Omega)$ with respect to the iterates of $P(x,D)$ and the wave front $WF_{s,\mathbb{P}}(u,P)$.

**Proposition 2.** *Let $u \in \mathfrak{D}'(\Omega)$ and $P(x,D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$, then the projection of $WF_{s,\mathbb{P}}(u,P)$ on $\Omega$ is the complementary of the biggest open subset $\Omega'$ of $\Omega$ where $u \in G^{s,\mathbb{P}}(\Omega', P)$.*

*Proof.* It follows the steps of the proofs of the classical theorems in the homogeneous case, see [2], and the quasihomogeneous case, see [16], and makes use essentially of the following lemma.

**Lemma 6.** *Let $u \in \mathfrak{D}'(\Omega)$ and $(x_0, \xi_0) \notin WF_{s,\mathbb{P}}(u, P)$, $U$ and $\Gamma$ be as in the definition 8, $K$ a compact neighbourhood of $x_0$ in $U$, $F$ be a $q$-quasiconic compact neighbourhood of $\xi_0$ in $\Gamma$ and $(\chi_N) \subset C_0^\infty(U)$ be a sequence equals to 1 on $K$ satisfying (2.2), then there exists $p_0 \in \mathbb{Z}_+, N_0 \in \mathbb{Z}_+$ such that the sequence $\left(\chi_{p_0 N + N_0} P^N u\right)$ satisfies jjj) in $F$.*

□

The microlocal property of the operator $P(x, D)$ with respect to the wave front $WF_{s,\mathbb{P}}(u, P)$ is the following result.

**Theorem 4.** *Let $u \in \mathfrak{D}'(\Omega)$ and $P(x, D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$, then*

$$WF_{s,\mathbb{P}}(u, P) \subset WF_{s,\mathbb{P}}(Pu) \subset WF_{s,\mathbb{P}}(u) \tag{3.2}$$

*Proof.* Suppose that $(x_0, \xi_0) \notin WF_{s,\mathbb{P}}(u)$, then there exists a neighbourhood $U$ of $x_0$, a $q$-quasiconic neighbourhood $\Gamma$ of $\xi_0$ and a bounded sequence $(u_N)$ in $\mathcal{E}'(\Omega)$ such that $u_N = u$ in $U$ and $|\widehat{u_N}(\xi)| \leq C\left(\frac{CN^s}{|\xi|_{\mathbb{P}}}\right)^{\mu N}$, $N = 1, 2, .., \ \xi \in \Gamma$. Let $K$ be a compact neighbourhood of $x_0$ in $U$, $F$ be a $q$-quasiconic compact neighbourhood of $\xi_0$ in $\Gamma$ and let $(\chi_N) \subset C_0^\infty(U)$ equal to 1 on $K$ satisfying (2.2). Choose $p \geq p_0 + N_0$ and set $f_N = \chi_{pN} P^N u$, we will show that this sequence satisfies jjj) since j) is true and jj) is fulfilled according to lemma 5.
We have

$$\widehat{f}_N(\xi) = \int e^{-i<x,\xi>} \chi_{pN} P^N u dx = \int u \, {}^t P^N \left(e^{-i<x,\xi>} \chi_{pN}\right) dx. \tag{3.3}$$

Set ${}^t P(x, D) = \sum\limits_{\alpha \in \mathbb{Z}_+^n \cap \mathbb{P}} a'_\alpha(x) D^\alpha$ and let $0 = k_0 < k_1 < .. < k_r = 1$, be the elements of the set $\left\{k = k(\alpha), \ \alpha \in \mathbb{Z}_+^n \cap \mathbb{P}\right\}$. Then

$${}^t P\left(e^{-i<x,\xi>} \chi_{pN+r}\right) = e^{-i<x,\xi>} |\xi|_{\mathbb{P}}^\mu R \chi_{pN+r} \ ,$$

where $R(x, \xi, D) = R_0 + ... + R_r$ and

$$R_l(x, \xi, D) = \sum_{\alpha \in \mathbb{Z}_+^n \cap \mathbb{P}} \sum_{\substack{\beta \leq \alpha \\ k(\beta) = k_l}} (-1)^{|\beta|} a'_\alpha(x) \frac{\xi^\beta}{|\xi|_{\mathbb{P}}^\mu} D^{\alpha - \beta} \ .$$

By iteration we find

$${}^t P^N \left(e^{-i<x,\xi>} \chi_{pN}\right) = e^{-i<x,\xi>} |\xi|_{\mathbb{P}}^{\mu N} R^N \chi_{pN} = e^{-i<x,\xi>} |\xi|_{\mathbb{P}}^{\mu N} \sum_{\substack{0 \leq l_i \leq r \\ 1 \leq i \leq N}} R_{l_1} ... R_{l_N} \chi_{pN} \ . \tag{3.4}$$

Since the coefficients of $R_l$ are in $G^{s,q}(\Omega)$, $\forall \xi \in \mathbb{R}^n$, then from lemma 4, we obtain for $<\alpha, a>\leq N$, $a \in \mathcal{A}$,

$$\left|D^\alpha R_{l_1}...R_{l_N}\chi_{pN+r}\right| \leq C_1^{N+1} N^{s\left(\mu<\alpha,a>+\mu^2 N-\sum\limits_{1\leq i\leq N}\mu^2 k_{l_i}\right)} \left(|\xi|_{\mathbb{P}}^{\mu}\right)^{\sum\limits_{1\leq i\leq N} k_{l_i}-N},$$

since $\left|\xi^\beta\right| \leq |\xi|_{\mathbb{P}}^{\mu k(\beta)}$, $\forall \beta \in \mathbb{Z}_+^n$. Then for $|\xi|_{\mathbb{P}} \geq N^{s\mu}, <\alpha, a>\leq N$, $a \in \mathcal{A}$, we get

$$\left|D^\alpha\left(R^N\chi_{pN}\right)\right| \leq C_2^{N+1} N^{s\mu<\alpha,a>}. \tag{3.5}$$

From $(3.3), (3.4), (3.5)$ and lemma 2, we obtain

$$\left|\widehat{f_N}(\xi)\right| = \left|\left(\left(R^N\chi_{pN}\right)u\right)\hat{}\,(\xi)\right| \leq C\left(CN^s\right)^{\mu N},\ \xi \in F, |\xi|_{\mathbb{P}} \geq N^{s\mu},$$

so $(x_0,\xi_0) \notin WF_{s,\mathbb{P}}(u,P)$, hence $WF_{s,\mathbb{P}}(u,P) \subset WF_{s,\mathbb{P}}(u)$.

Since

$$WF_{s,\mathbb{P}}(u,P) = WF_{s,\mathbb{P}}(Pu,P) \subset WF_{s,\mathbb{P}}(Pu)$$

and $WF_{s,\mathbb{P}}(Pu) \subset WF_{s,\mathbb{P}}(u)$, according to theorem 3, so the proof of theorem 4 is complete. □

## 4. The multi-anisotropic Gevrey microlocal regularity

We obtain in this section a result of Gevrey microlocal regularity for a class of multi-anisotropic hypoelliptic differential operators characterized by the following definition.

**Definition 9.** *Let $x_0 \in \Omega, \xi_0 \in \mathbb{R}^n\setminus\{0\}$ and $P(x,D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$, we denote $(x_0,\xi_0) \notin \sum_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}(P)$ if there exists an open neighbourhood $U \subset \Omega$ of $x_0$, an open $q$-quasiconic neighbourhood $\Gamma \subset \mathbb{R}^n\setminus\{0\}$ of $\xi_0$ and a constant $c > 0$ such that $\forall (x,\xi) \in U \times \Gamma$,*

$$\begin{cases} |\xi|_{\mathbb{P}}^{\mu'} \leq c\,|P(x,\xi)|, \\ \left|D_x^\alpha D_\xi^\beta P(x,\xi)\right| \leq c^{|\alpha|+1} <\alpha, q>^{s\mu<\alpha,q>} |P(x,\xi)|\,|\xi|_{\mathbb{P}}^{\delta|\alpha|-\rho|\beta|}, \end{cases} \tag{4.1}$$

*where the numbers $\rho, \delta, \mu'$ and $\mu$ satisfy* $0 \leq \delta < \rho \leq 1$ *and* $\delta\mu < \mu' \leq \mu$.

We need the following lemma which is a modification of the similar result of [2, lemme 3.8].

**Lemma 7.** *Under the notations of definition 9, if $\chi_N \in C_0^\infty(U)$ satisfies $(2.2)$, so there exists $C > 0$ such that for $(x,\xi) \in U \times \Gamma, h_1, .., h_j \in$*

$\mathbb{Z}_+, a\in\mathcal{A}, <\alpha^1+..+\alpha^j,a>\leq N,\ \beta^1,..,\beta^{j-1}\in\mathbb{Z}_+^n$ :

$$\left|D^{\alpha^1}P^{h_1}P^{\left(\beta^1\right)}...D^{\alpha^{j-1}}P^{h_{j-1}}P^{\left(\beta^{j-1}\right)}D^{\alpha^j}P^{h_j}\chi_N\right|$$
$$\leq\ C^{N+1+|h_1|+..+|h_j|}<\alpha,a>^{s\mu<\alpha,a>}\left|P\left(x,\xi\right)\right|^{h_1+..+h_j+j-1}\left|\xi\right|_{\mathbb{P}}^{\delta|\alpha|-\rho|\beta|},$$

*where* $\alpha=\alpha^1+..+\alpha^j,\ \beta=\beta^1+..+\beta^{j-1}$.

The principal result of this work is the following theorem.

**Theorem 5.** *Let* $\Omega$ *be an open subset of* $\mathbb{R}^n$*,* $u\in\mathfrak{D}'\left(\Omega\right)$ *and* $P(x,D)$ *be a differential operator with coefficients in* $G^{s,q}\left(\Omega\right)$ *and let* $\rho,\delta,\mu'$ *and* $\mu$ *be real numbers satisfying* $0\leq\delta<\rho\leq1$ *and* $\delta\mu<\mu'\leq\mu$*, then*

$$WF_{s',\mathbb{P}}\left(u\right)\subset WF_{s,\mathbb{P}}\left(u,P\right)\cup\sum\nolimits_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}\left(P\right),\tag{4.2}$$

*where* $s'=\max\left(\frac{s\mu}{\mu'-\delta\mu},\frac{s}{\rho-\delta}\right)$.

*Proof.* Let $(x_0,\xi_0)\notin WF_{s,\mathbb{P}}\left(u,P\right)\cup\sum_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}\left(P\right)$, then there exists $C>0, M\in\mathbb{R}$, a neighbourhood $U$ of $x_0$ in $\Omega$, a $q$-quasiconic neighbourhood $\Gamma$ of $\xi_0$ in $\mathbb{R}^n\backslash\left\{0\right\}$ and a sequence $\left(f_N\right)\subset\mathcal{E}'\left(\Omega\right)$ such that the conditions j),jj) and jjj) of definition 8 are fulfilled. Let $K$ be a compact neighbourhood of $x_0$ in $U$, $F$ a $q-$quasiconic compact neighbourhood of $\xi_0$ in $\Gamma$ such that (4.1) is hold and let $\chi_N\in C_0^\infty\left(U\right),\ \chi_N=1$ on $K$ satisfying (2.2) and $p$ a large enough integer. Set $u_N=\chi_{pN}u$ and let's prove that this sequence satisfies iii) since i) and ii) are fulfilled. We write

$${}^tP\left(e^{-i<x,\xi>}w\right)=e^{-i<x,\xi>}\left({}^tP\left(x,-\xi\right)\left(I-R\right)\right)w,$$

where

$$-R\left(x,\xi,D\right)=\sum_{\beta\neq0}\frac{1}{\beta!}\,\frac{{}^tP^{(\beta)}\left(x,-\xi\right)}{{}^tP\left(x,-\xi\right)}D^\beta.$$

By iteration we get

$${}^tP^N\left(e^{-i<x,\xi>}w\right)=e^{-i<x,\xi>}\left({}^tP\left(x,-\xi\right)\left(I-R\right)\right)^Nw.$$

The fact that we can divide by ${}^tP\left(x,-\xi\right)$ is due to the following lemma which can easily be proved.

**Lemma 8.** *If* $(x_0,\xi_0)\notin\sum_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}(P)$*, then* $(x_0,-\xi_0)\notin\sum_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}({}^tP)$.

Set

$$w_N=\sum_{h_1+..+h_N\leq\left[\frac{\mu'-\delta\mu}{\rho-\delta}N\right]}R^{h_1}\left({}^tP\right)^{-1}...R^{h_N}\left({}^tP\right)^{-1}\chi_{pN},$$

where ${}^tP = {}^tP\,(x,-\xi)\,.$ Then this function satisfies

$$\left({}^tP\,(I-R)\right)^N w_N = \chi_{pN} - e_N,$$

where

$$e_N = \sum_{j=1}^{N} \left({}^tP\,(I-R_N)\right)^{N-j} \sum_{h_j+..+h_N=\left[\frac{\mu'-\delta\mu}{\rho-\delta}N\right]} {}^tPR^{h_j+1}\left({}^tP\right)^{-1}R^{h_{j+1}}...R^{h_N}\left({}^tP\right)^{-1}\chi_{pN}.$$

Hence

$$\widehat{u}_N\,(\xi) = \widehat{w_N f_N}\,(\xi) + \widehat{e_N u}\,(\xi)\,,\ \xi\in F. \tag{4.3}$$

We will estimate both terms of the second member of (4.3). Let $a\in\mathcal{A}$ and $0=k_0<k_1<...<k_r=1$, be the elements of the set $\left\{k=<\alpha,a>,\alpha\in\mathbb{Z}_+^n\cap\mathbb{P}\right\}$. we write $R=R_1+...+R_r$ where

$$-R_l\,(x,\xi,D) = \sum_{<\beta,a>=k_l} \frac{1}{\beta!}\,\frac{{}^tP^{(\beta)}\,(x,-\xi)}{{}^tP\,(x,-\xi)}D^\beta,$$

then we have

$$w_N = \sum_{h_1+..+h_N\leq\left[\frac{\mu'-\delta\mu}{\rho-\delta}N\right]}\ \sum_{\substack{1\leq 1_j\leq r\\ 1\leq j\leq h_1}} ... \sum_{\substack{1\leq N_j\leq r\\ 1\leq j\leq h_N}} \left(R_{1_1}...R_{1_{h_1}}\right)\left({}^tP\right)^{-1}...\left(R_{N_1}...R_{N_{h_N}}\right)\left({}^tP\right)^{-1}\chi_{pN}.$$

Since $<\alpha,a>\leq|\alpha|\leq\mu<\alpha,a>$, so from lemma 7, we have for $<\alpha,a>\leq N, a\in\mathcal{A}$,

$$\begin{aligned}&\left|D^\alpha\left(R_{1_1}...R_{1_{h_1}}\right)\left({}^tP\right)^{-1}...\left(R_{N_1}...R_{N_{h_N}}\right)\left({}^tP\right)^{-1}\chi_{pN}\right|\\ \leq\ & C^{N+1}N^{s\mu\left(<\alpha,a>+\sum^* k_{l_i}\right)}\left|P\,(x,\xi)\right|^{-N}|\xi|_{\mathbb{P}}^{(\delta-\rho)\sum^* k_{l_i}+\mu\delta<\alpha,a>},\end{aligned}$$

where $\sum^*$ means the sum over $1\leq l\leq N, 1\leq i\leq h_l, 1\leq l_i\leq r$. Since the number of terms in the sum $w_N$ is bounded from above by $C_0^N$, so $\exists C>0$ such that, for $<\alpha,a>\leq N,\xi\in F,|\xi|_{\mathbb{P}}^{\rho-\delta}\geq N^{s\mu}$, we have

$$\left|D^\alpha w_N\right|\leq C_1^{N+1}N^{s\mu<\alpha,a>}|\xi|_{\mathbb{P}}^{(\mu\delta-\mu')N},$$

from lemma 6, we obtain for $|\xi|_{\mathbb{P}}^{\rho-\delta}\geq N^{s\mu}$,

$$\begin{aligned}\left|\widehat{w_N f_N}\,(\xi)\right| &\leq C_1\left(C_1N^s\right)^{\mu N}|\xi|^M\,|\xi|_{\mathbb{P}}^{(\mu\delta-\mu')N}\\ &\leq C_2\left(\frac{C_2N^{\frac{s\mu}{\mu'-\delta\mu}}}{|\xi|_{\mathbb{P}}}\right)^{\frac{(\mu\delta-\mu')}{\mu}\mu N}|\xi|^M.\end{aligned} \tag{4.4}$$

By the same procedure in the estimate of $w_N$, we get for $e_N$,

$$|D^\alpha e_N| \leq C_3^{N+1} N^{s\mu<\alpha,a>} \left( \frac{N^{\frac{s}{\rho-\delta}}}{|\xi|_{\mathbb{P}}} \right)^{(\rho-\delta)\mu N}, < \alpha, a > \leq N, |\xi|_{\mathbb{P}}^{\rho-\delta} \geq N^{s\mu}.$$

Let $M_1$ be the order of the distribution $u$ in $K$, so

$$|\widehat{ue_N}(\xi)| \leq C_3'^{N+1} |\xi|^{M_1} \left( \frac{N^{\frac{s}{\rho-\delta}}}{|\xi|_{\mathbb{P}}} \right)^{(\rho-\delta)\mu N}. \tag{4.5}$$

From $(4.3), (4.4)$ and $(4.5)$ we easily obtain that

$$(x_0, \xi_0) \notin WF_{s',\mathbb{P}}(u).$$

$\square$

## 5. Consequences

This section gives some corollaries of the obtained result.

**Corollary 1.** *If $P(x, D)$ is a differential operator with analytic coefficients, satisfying (4.1) with $|\xi|_{\mathbb{P}} = |\xi|$, then theorem 5 coincides with the principal theorem 5.1 of Bolley-Camus* [2]*, i.e. $\forall s \geq 1$,*

$$WF_s(u) \subset WF_s(u, P) \cup \sum\nolimits_{\rho,\delta,s}^{\mu,\mu',\mathbb{P}}(P).$$

**Remark 5.** *The results of* [7] *and* [10] *can be included in this corollary.*

**Corollary 2.** *If the differential operator $P(x, D)$ is $q$-quasihomogeneous with coefficients in $G^{s,q}(\Omega)$, then $|\xi|_{\mathbb{P}} = |\xi|_q$ and*

$$\sum\nolimits_{1,0,s}^{\mu,\mu,\mathbb{P}}(P) = \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} : P_q(x, \xi) = 0\},$$

*where $P_q(x, \xi)$ is the principal $q$- quasihomogeneous part of $P(x, \xi)$. Consequently theorem 5 coincides with the principal theorem of* [16]*, i.e. $\forall s \geq 1$*

$$WF_{s,q}(u) \subset WF_{s,q}(u, P) \cup \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} : P_q(x, \xi) = 0\}.$$

**Definition 10.** *The operator $P(x, D)$ is said multi-quasielliptic in $\Omega$, if it is regular and $\forall x_0 \in \Omega$,*

$$\exists C > 0,\ \exists R \geq 0,\ (|\xi|_{\mathbb{P}})^{\mu(\mathbb{P})} \leq C |P(x_0, \xi)|,\ \forall \xi \in \mathbb{R}^n, |\xi| \geq R.$$

The multi-anisotropic Gevrey regularity of the solutions of multi-quasielliptic differential equations, see [17] and [4], is obtained easily from the following microlocal result.

**Corollary 3.** *Let* $u \in \mathfrak{D}'(\Omega)$ *and* $P(x,D)$ *be a multi-quasielliptic differential operator with coefficients in* $G^{s,q}(\Omega)$, *then* $\forall s \geq 1$, *we have*

$$WF_{s,\mathbb{P}}(u) = WF_{s,\mathbb{P}}(u,P) = WF_{s,\mathbb{P}}(Pu).$$

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